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DERIVATION OF EQUATIONS FOR A LAYER OF VARIABLE THICKNESS BASED ON EXPANSIONS IN TERMS OF LEGENDRE'S POLYNOMIALS

A. E. Alekseev

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Equations for the deformation of an elastic layer of variable thickness for small elongations, displacements, and rotations are generally represented as three-dimensional equations of the linear theory of elasticity. Different approaches to the reduction of the dimensions of the initial problem are based on the fact that the layer thickness is small in comparison with the other linear sizes. One approach uses the method of expansion of the sought functions in segments of the series in terms of Legendre's polynomials (for example, [1]).

A method for reducing the dimensions of the problems of elastic deformations of plates and shells of constant thickness with arbitrary conditions for displacements and stresses on the face surfaces is presented in [2]. It is based on several approximations of the same unknown functions in the form of segments of Legendre's polynomial series. Developing this approach, the author [3] presented the equations for the deformation of a layer of variable thickness in an arbitrary curvilinear coordinate system.

A method of reducing three-dimensional equations of the theory of elasticity to a sequence of two-dimensional problems of the theory of an elastic layer, which is the development of the results of [2, 3], is presented below.

1. Definition of a Layer. In continuum mechanics the term "shell" denotes a body bounded by two curvilinear surfaces with the distance between them small in comparison with the other dimensions of the body.

Denote by V the part of the three-dimensional space R^3 occupied by the shell. Define the position of the face surfaces S^+ and S^- by the radius-vectors $\mathbf{R}^+$ and $\mathbf{R}^-$ as functions of the Gaussian coordinates ξ^α:

$$\mathbf{R}^+ = \mathbf{R}^+(\xi^\alpha),\ \mathbf{R}^- = \mathbf{R}^-(\xi^\alpha),\ \{\xi^\alpha\} \in S_\xi \subset R^2.$$

Hereafter, Greek indices have the range 1, 2, and Latin indices have the range 1, 2, 3.

The functions $\mathbf{R}^+$, $\mathbf{R}^-$ map a plane area S_ξ with boundary L_ξ in R^2 onto the face surfaces S^+, S^-, respectively. The position of each internal point of the shell V is defined by the vector function of curvilinear coordinates ξ^k:

$$\mathbf{R}(\xi^k) = \mathbf{r}_0(\xi^\alpha) + \xi^3\Delta\mathbf{r}(\xi^\alpha),\ \{\xi^k\} \in V_\xi \subset R^3. \tag{1.1}$$

Here

$$V_\xi = \{\xi^k | \xi^\alpha \in S_\xi \subset R^2,\ \xi^3 \in [-1, 1]\},$$
$$\mathbf{r}_0 = 0{,}5(\mathbf{R}^+(\xi^\alpha) + \mathbf{R}^-(\xi^\alpha)),\ \Delta\mathbf{r} = 0{,}5(\mathbf{R}^+(\xi^\alpha) - \mathbf{R}^-(\xi^\alpha)). \tag{1.2}$$

In this case the vector function $\mathbf{R}$ maps V_ξ onto V, and the vector-function $\mathbf{r}_0$ maps a plane area S_ξ onto a surface S_0 in three-dimensional space which is called from here on a middle surface.

Let h denote half a thickness of a layer along ξ^3. We obtain from (1.2)

$$h = (\Delta\mathbf{r} \cdot \Delta\mathbf{r})^{0{,}5},\ \Delta\mathbf{r} = h\mathbf{n}$$

(here $\mathbf{n}$ is the unit vector along ξ^3).

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Let Σ be a side surface of the shell, and let L be the line of intersection of Σ and S_0. Then, according to (1.1), Σ is a ruled surface formed by a family of straight lines passing through the points of L along **n**. Fix a point with coordinates $\{\xi_0^\alpha\}$ on S_ξ. Then, due to (1.1), the vector-function **R** associates this point with a segment of a straight line in three-dimensional space ending on the face surfaces. Moreover, if the point belongs to L, then the entire segment belongs to the side surface Σ.

So, the geometry of a shell of variable thickness is completely specified by the vector-functions $\mathbf{R}^+$, $\mathbf{R}^-$. In this case **n** is not necessarily normal to the surface S_0, so the side surface Σ and the middle surface S_0 may not intersect at right angle. From here on a shell of variable thickness whose geometry is based on the relationship (1.1) is called the layer of variable thickness, or simply, the layer.

2. Local Bases of a Coordinate System of the Layer. According to (1.1), it is possible to take a triple of numbers $\{\xi^k\}$ as the coordinates of each point of the layer V, where $\{\xi^\alpha\}$ are the Gaussian coordinates on the middle surface S_0, and $\xi^3 \in [-1, 1]$ is the coordinate along **n**. Such a curvilinear coordinate system is called a coordinate system of the layer.

Differentiating both parts of the equality (1.1) with respect to the variables ξ^k we obtain the vector-functions

$$\mathbf{э}_\alpha = \mathbf{R}_{,\alpha} = \left(\mathbf{R}_{,\alpha} = \frac{\partial \mathbf{R}}{\partial \xi^\alpha}\right) = \mathbf{r}_{0,\alpha} + \Delta \mathbf{r}_{,\alpha}\xi^3, \quad \mathbf{э}_3 = \mathbf{R}_{,3} = \Delta \mathbf{r} = h\mathbf{n}, \tag{2.1}$$

which form a covariant local basis for the coordinate system of the layer. The corresponding biorthogonal (contravariant) local basis, consisting of the vector functions $\mathbf{э}^i$, is defined by the

$$\mathbf{э}^i \cdot \mathbf{э}_j = \delta^i_j$$

(δ^i_j is the Kronecker symbol) as

$$\mathbf{э}^1 = \frac{\mathbf{э}_2 \times \mathbf{э}_3}{J}, \ \mathbf{э}^2 = \frac{\mathbf{э}_3 \times \mathbf{э}_1}{J}, \ \mathbf{э}^3 = \frac{\mathbf{э}_1 \times \mathbf{э}_2}{J}, \quad J = \mathbf{э}_1 \cdot (\mathbf{э}_2 \times \mathbf{э}_3). \tag{2.2}$$

Let us denote

$$\overset{0}{\mathbf{э}}_\alpha = \mathbf{r}_{0,\alpha} = \mathbf{э}_\alpha(\xi^\beta, 0), \ \overset{0}{\mathbf{э}}_3 = \mathbf{э}_3(\xi^\alpha), \ \mathbf{э}^{0\alpha} = \mathbf{э}^i(\xi^\alpha, 0).$$

The triples of the vectors $\overset{0}{\mathbf{э}}_i$ and $\mathbf{э}^{0i}$ form local bases on the middle surface of the layer. The local bases at an arbitrary point of the layer are determined by the parallel transfer from the proper point of the middle surface. For each point of the layer we define a triple of vectors $\mathbf{a}_i$:

$$\mathbf{a}_\alpha = \mathbf{n} \times (\overset{0}{\mathbf{э}}_\alpha \times \mathbf{n}), \ \mathbf{a}_3 = \mathbf{n}, \tag{2.3}$$

and consider it further as the main local basis of the layer (covariant). Following the formulas from vector algebra, we obtain from (2.3)

$$\mathbf{a}_2 = \overset{0}{\mathbf{э}}_2 - \mathbf{n}\cdot(\overset{0}{\mathbf{э}}_\alpha \cdot \mathbf{n})$$

i.e., the vector $\mathbf{a}_\alpha$ is the projection of $\overset{0}{\mathbf{э}}_2$ onto the plane orthogonal to the unit vector **n**. The corresponding biorthogonal (contravariant) basis is found from the conditions

$$\mathbf{a}_i \cdot \mathbf{a}^j = \delta^j_i$$

and is of the form

$$\mathbf{a}^\alpha = \mathbf{э}^{0\alpha}, \ \mathbf{a}^3 = \mathbf{n}.$$

It is clear from (2.3) that **n** is orthogonal to the base vectors $\mathbf{a}^\alpha$, $\mathbf{a}_\alpha$.

Thus, as is given above, we can define three types of local bases of a curvilinear coordinate system of the layer: $\mathbf{э}_i$, $\overset{0}{\mathbf{э}}_i$, $\mathbf{a}_i$. Every vector from one basis can be represented as a linear combination of vectors of another basis. Denote

$$g_{ij} = э_i \cdot э_j,\ g^0_{ij} = э^0_i \cdot э^0_j,\ a_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j, \tag{2.4}$$

where g_{ij} are the components of a metric tensor of the coordinate system ξ^i. It follows from (2.1) and (2.4) that

$$g_{33} = h^2,$$

and from (2.3) we have

$$a_{33} = 1,\ a_{32} = 0.$$

Considering the notation

$$g_\alpha = g_{\alpha 3}/h,\ b^\beta_\alpha = -(\mathbf{a}_{3,\alpha} \cdot a^\beta)$$

we write formula (2.1) as

$$э_\alpha = m^\beta_\alpha \mathbf{a}_\beta + g_\alpha \mathbf{a}_3,\ э_3 = h\mathbf{a}_3, \tag{2.5}$$

where

$$m^\beta_\alpha = \delta^\beta_\alpha - b^\beta_\alpha h\xi^3. \tag{2.6}$$

For the components of a metric tensor g_{ij} we have, respectively,

$$g_{\alpha\beta} = m^\gamma_\alpha m^\lambda_\beta a_{\gamma\lambda} + g_\alpha g_\beta,\ g_{\alpha 3} = hg_\alpha,\ g_{33} = h^2. \tag{2.7}$$

From (2.7) we obtain formulas connecting the determinants g, g^0, a of the matrices $\| g_{ij} \|$, $\| \overset{0}{g}_{ij} \|$, $\| a_{ij} \|$:

$$g = h^2 m^2 a,\ g^0 = h^2 a$$

($m = m^1_1 m^2_2 - m^2_1 m^1_2$ is the determinant of the matrix $\| m_{\alpha\beta} \|$. Following [4], we can write, due to (2.6), the expression for m as a polynomial of the second degree in the variable ξ^3:

$$m = 1 - 2Hh\xi^3 + Kh^2(\xi^3)^2$$
$$(2H = b^\alpha_\alpha,\ K = b^1_1 b^2_2 - b^2_1 b^1_2).$$

The relationship between the contravariant basis of the layer $э^i$ and the main local basis $\mathbf{a}_i$ is of the form

$$э^\beta = \mathbf{a}_\alpha (m^{-1})^{\alpha\beta},\ э^3 = (\mathbf{a}^3 - g_\gamma \mathbf{a}_\alpha (m^{-1})^{\alpha\gamma})/h \tag{2.8}$$

($\| (m^{-1})^{\alpha\beta} \|$ is the inverse matrix of $\| m^{\alpha\beta} \|$).

The validity of the equalities (2.8) can be verified with the help of the relationships (2.2) and the formulas (2.5). The components of the inverse matrix $\| (m^{-1})^{\alpha\beta} \|$ are represented in the form [4]

$$(m^{-1})^\beta_\alpha = (\delta^\beta_\alpha + h\xi^3(b^\beta_\alpha - 2H\delta^\beta_\alpha))/m.$$

3. Equations of the Linear Theory of Elasticity in an Arbitrary Curvilinear Coordinate System. Consider an arbitrary curvilinear coordinate system ξ^i. The equations for equilibrium of the continuum are written in vector form [1] as

$$\hat{t}^i_{,i} + \hat{f} = 0,\ \hat{t}^i = J\, t^i,\ \hat{f} = J\, f,\ t^i = \sigma^{ij} э_j; \tag{3.1}$$

$$э_i \times \hat{t}^i = 0, \tag{3.2}$$

where J is the Jacobian of the transformation of the coordinates; σ^{ij} are the components of the stress tensor; **f** is the vector of volume forces. Equation (3.2) is the condition for symmetricity of the stress tensor.

Since only small elongations, displacements, and rotation angles are considered, we restrict ourselves to the linear dependences of the components of the strain tensor ε_{ij} on the displacement vector **u**:

$$2\varepsilon_{ij} = (\mathfrak{s}_i \cdot \mathbf{u}_{,j}) + (\mathfrak{s}_j \cdot \mathbf{u}_{,i}). \tag{3.3}$$

The generalized Hook's law for the stress-strain relations is of the form

$$\sigma^{ij} = C^{ijks}\varepsilon_{ks} \tag{3.4}$$

(C^{ijks} are the contravariant components of the 4th rank tensor defining the properties of elastic media).

It is convenient to write the relations (3.4) in vector form:

$$\hat{\mathbf{t}}^i = J\tilde{C}^{ij} \cdot \mathbf{u}_{,j}. \tag{3.5}$$

Here $\tilde{C}^{ij}$ is the operator given by the formula

$$\tilde{C}^{ij} = C^{ijks}(\mathfrak{s}_k{}^* \mathfrak{s}_s),$$

where * is the symbol for tensor multiplication.

For simplicity we restrict further discussion to the case where the boundary S of the deforming body consists of two parts:

S_u (the displacements are specified)

$$\mathbf{u}|_{S_u} = \mathbf{u}_*; \tag{3.6}$$

S_σ (the stresses are specified):

$$\hat{\mathbf{t}}^i\nu_i|_{S_\sigma} = \mathbf{P}_* \tag{3.7}$$

(ν_i are cosines of the outward normal vector to the boundary S; and $\mathbf{u}_*$, $\mathbf{P}_*$ are the specified vector functions on S).

Equations (3.1), (3.5) and the boundary conditions (3.6), (3.7) define a boundary problem of the linear theory of elasticity.

4. Expansion of Functions in Legendre's Polynomials. Select the coordinate system ξ^k of the layer as a curvilinear coordinate system. In this case the coordinate ξ^3 belongs to $[-1, 1]$, and the unknown functions $\mathbf{u}$, $\hat{\mathbf{t}}^i$ can be represented as the Legendre's polynomial series

$$\mathbf{u} = \sum_{k=0}^{\infty} [\mathbf{u}]^k P_k, \quad \hat{\mathbf{t}}^i = \sum_{k=0}^{\infty} [\hat{\mathbf{t}}^i]^k P_k. \tag{4.1}$$

Here $P_k(\xi^3)$ are the orthogonal Legendre's polynomials; $[\mathbf{u}]^k$, $[\hat{\mathbf{t}}^i]^k$ are the expansion coefficients dependent on the Gaussian coordinates $\{\xi^k\} \in S_\xi \subset R^2$:

$$[\mathbf{u}]^k = \frac{(1 + 2k)}{2} \int_{-1}^{1} \mathbf{u} P_k d\xi^3, \quad [\hat{\mathbf{t}}^i]^k = \frac{(1 + 2k)}{2} \int_{-1}^{1} \hat{\mathbf{t}}^i P_k d\xi^3.$$

Expand the quantities $\hat{\mathbf{t}}^i$ in the main local basis $\mathbf{a}_i$. According to (2.5), (3.1) we have

$$\hat{\mathbf{t}}^i = J\sigma^{ij}\mathfrak{s}_j = \sqrt{a}\,mh(\sigma^{i\alpha}\mathfrak{s}_\alpha + \sigma^{i3}\mathfrak{s}_3) = \sqrt{a}\,mh(\sigma^{i\alpha}m^\beta_\alpha \mathbf{a}_\beta + \sigma^{ik}g_{k3}\mathbf{a}_3),$$

Using the rule of index lowering, we get

$$\hat{\mathbf{t}}^i = \sqrt{a}\,mh(\sigma^{i\alpha}m^\beta_\alpha \mathbf{a}_\beta + \sigma^i_3 \mathbf{n}/h). \tag{4.2}$$

Substituting (4.2) into formulas (4.1) for the expansion of $\hat{\mathbf{t}}^i$ in Legendre's polynomials, after simple transformations we have

$$\dot{t}^\alpha = \sqrt{a}h \sum_{k=0}^{\infty} \frac{1+2k}{2} (\overset{(k)}{M}{}^{\alpha\gamma} \mathbf{a}_\gamma + \overset{(k)}{Q}{}^\alpha \mathbf{n}) P_k, \tag{4.3}$$

where

$$\overset{(k)}{M}{}^{\alpha\gamma} = \int_{-1}^{1} m m^\gamma_\beta \sigma^{\alpha\beta} P_k d\xi^3; \quad \overset{(k)}{Q}{}^\alpha = \int_{-1}^{1} m \sigma^\alpha_3 P_k d\xi^3 / h.$$

The quantities $\overset{(k)}{M}{}^{\alpha\beta}$ are called moments of the tangential stresses of the k-th order, and $\overset{(k)}{Q}{}^\alpha$ are called moments of the transverse (shearing) forces of the k-th order.

Let $\mathbf{N}^\alpha$ and $\mathbf{M}^\alpha$ stand for the vectors of forces and moments, which act on an area $\xi^\alpha = \text{const}$ and are related to a layer of thickness 2h. By definition we have

$$\mathbf{N}^\alpha = \int_{-1}^{1} \dot{t}^\alpha d\xi^3 / 2J^0, \quad \mathbf{M}^\alpha = \int_{-1}^{1} (\text{э}_3 \times t^\alpha) \xi^3 d\xi^3 / 2J^0. \tag{4.4}$$

Substituting (4.3) into (4.4) we obtain

$$2\mathbf{N}^\alpha = \overset{(0)}{M}{}^{\alpha\gamma} \mathbf{a}_\gamma + \overset{(0)}{Q}{}^\alpha \mathbf{n}, \quad 2\mathbf{M}^\alpha = (\mathbf{n} \times \mathbf{a}_\gamma) h \overset{(1)}{M}{}^{\alpha\gamma}. \tag{4.5}$$

It follows from (4.5) that the first terms of the expansion of stresses in the Legendre's polynomial series have an important property — they can be considered as forces and moments acting on an element of the layer.

As in the case of forces, we expand the displacement vector **u** from (4.1) in the main basis:

$$\mathbf{u} = \sum_{k=0}^{\infty} (\overset{(k)}{U}{}^\gamma \mathbf{a}_\gamma + \overset{(k)}{W} \mathbf{n}) P_k. \tag{4.6}$$

Here

$$\overset{(k)}{U}{}^\gamma = \frac{1+2k}{2} \int_{-1}^{1} (\mathbf{u} \cdot \mathbf{a}^\gamma) P_k d\xi^3, \quad \overset{(k)}{W} = \frac{1+2k}{2} \int_{-1}^{1} (\mathbf{u} \cdot \mathbf{n}) P_k d\xi^3 \tag{4.7}$$

are the moments of tangential and transverse displacements of the k-th order.

Denote an arbitrary rigid displacement by $\mathbf{u}_*$. Then for an arbitrary point of the layer with radius-vector **R** the equality holds [5] in the case of small shifts.

$$\mathbf{u}_* = \mathbf{v}_0 + \mathbf{R} \times \omega_0, \tag{4.8}$$

Here $\mathbf{v}_0$, ω_0 are arbitrary constants describing the displacement of an origin and rotation.

By the definition of geometry of the layer, the radius-vector **R** is found from formula (1.1). Substituting this expression into (4.8) we get

$$\mathbf{u}_* = \mathbf{v}_* + h(\mathbf{n} \times \omega_0)\xi^3, \quad \mathbf{v}_* = \mathbf{v}_0 + \mathbf{r}_0 \times \omega_0. \tag{4.9}$$

Let us find the terms of the expansion (4.6) with the rigid displacement involved. To do this we substitute (4.9) into the formulas (4.7). After integrating with regard to the properties of Legendre's polynomials we have

$$\begin{gathered} \overset{(0)}{U}{}^\gamma = \mathbf{v}_* \cdot \mathbf{a}^\gamma, \quad \overset{(1)}{U}{}^\gamma = (\mathbf{n} \times \omega_0), \quad \overset{(k)}{U}{}^\gamma = 0 \ (k = \overline{2, \infty}), \\ \overset{(0)}{W} = \mathbf{v}_* \cdot \mathbf{n}, \quad \overset{(k)}{W} = 0 \ (k = \overline{1, \infty}). \end{gathered} \tag{4.10}$$

Hence, it follows from (4.10) that an arbitrary rigid displacement of the layer appears in the first terms $\overset{(0)}{U}{}^{\alpha}$, $\overset{(1)}{U}{}^{\alpha}$, $\overset{(0)}{W}$ of the expansions of displacements in Legendre's polynomials. This fact has not only an important physical meaning, but it is a natural restriction on the minimum number of terms in approximating segments of the series (4.1) also. Thus, if the moment state is taken into account, the number of terms in approximating segments of the series (4.1) for displacements should not be less than two for tangential shifts and less than one for transverse ones.

Approximating the quantities $\hat{\mathbf{t}}^i$, $\mathbf{u}$ in the form of the series (4.1) we consider the expansion coefficients dependent on the two variables (the Gaussian coordinates on the middle surface) as unknown. However, the decrease in number of independent variables is attained at the cost of the increase in number of unknowns to infinity. So the next step is to truncate the series (4.1) and to reduce the initial differential problem to the solution of a finite system of equations in two independent variables.

5. Approximations of Stresses. Consider the equations of equilibrium of the continuous medium in the form equivalent to (3.1)

$$\mathbf{n}\times(\hat{\mathbf{t}}^i_{,i}+\hat{\mathbf{f}})=0,\ \mathbf{n}\cdot(\hat{\mathbf{t}}^i_{,i}+\hat{\mathbf{f}})=0, \tag{5.1}$$

where, as above, $\mathbf{n}$ is the unit vector along the ξ^3-axis.

The vector $\mathbf{n}$ does not depend on the variable ξ^3. So, expanding the equalities (5.1) in Legendre's polynomial series, we obtain a system of equations for every $N \geq 0$

$$\begin{gathered}\mathbf{n}\times([\hat{t}^{\alpha}]^k_{,\alpha}+[\hat{t}^3_{,3}]^k+[\hat{f}]^k)=0\ (k=\overline{0,N+1}),\\ \mathbf{n}\cdot([\hat{t}^{\alpha}]^k_{,\alpha}+[\hat{t}^3_{,3}]^k+[\hat{f}]^k)=0\ (k=\overline{0,N}).\end{gathered} \tag{5.2}$$

We multiply the equalities (5.2) by P_k for every k and sum. The result is the following:

$$\begin{gathered}\mathbf{n}\times\hat{\mathbf{T}}'^{\alpha}_{,\alpha}+\mathbf{n}\times\sum_{k=0}^{N+1}[\hat{\mathbf{t}}^3_{,3}]^kP_k+\mathbf{n}\times\hat{\mathbf{F}}=0,\\ \mathbf{n}\cdot\hat{\mathbf{T}}''^{\alpha}_{,\alpha}+\mathbf{n}\cdot\sum_{k=0}^{N}[\hat{\mathbf{t}}^3_{,3}]^kP_k+\mathbf{n}\cdot\hat{\mathbf{F}}=0.\end{gathered} \tag{5.3}$$

Here the quantities $\hat{\mathbf{T}}'^{\alpha}\hat{\mathbf{T}}''^{\alpha}$, $\mathbf{F}$ stand for the segments of the series

$$\begin{gathered}\hat{\mathbf{T}}'^{\alpha}=\sum_{k=0}^{N+1}[\hat{\mathbf{t}}^{\alpha}]^kP_k,\ \hat{\mathbf{T}}''^{\alpha}=\sum_{k=0}^{N}[\hat{\mathbf{t}}^{\alpha}]^kP_k,\\ \hat{\mathbf{F}}=\mathbf{n}\times(\sum_{k=0}^{N+1}[\hat{\mathbf{f}}]^k\times\mathbf{n}P_k)+\mathbf{n}\cdot(\sum_{k=0}^{N}[\hat{\mathbf{f}}]^k\cdot\mathbf{n}P_k).\end{gathered} \tag{5.4}$$

Consider a function $a(\xi)$ and the corresponding segment of a series $A(\xi)=\sum_{k=0}^{Q}[a]^kP_k(\xi)$. Then for a derivative $A_{,\xi}$ the following equality holds:

$$A_{,\xi}=\sum_{k=0}^{Q-1}[a_{,\xi}]^kP_k.$$

Using this property we conclusively obtain from (5.3)

$$\mathbf{n}\times\hat{\mathbf{T}}'^i_{,i}+\mathbf{n}\times\hat{\mathbf{F}}=0,\ \mathbf{n}\cdot\hat{\mathbf{T}}''^i_{,i}+\mathbf{n}\cdot\hat{\mathbf{F}}=0, \tag{5.5}$$

where, for short, the following notation is introduced:

$$\hat{\mathbf{T}}'^3=\hat{\mathbf{T}}''^3=\hat{\mathbf{T}}^3=\mathbf{n}\times(\sum_{k=0}^{N+2}\{[\hat{\mathbf{t}}^3]^kP_k\}\times\mathbf{n})+\mathbf{n}\cdot(\sum_{k=0}^{N+1}\{[\hat{\mathbf{t}}^3]^kP_k\}\cdot\mathbf{n}). \tag{5.6}$$

Thus, in Eqs. (5.5) we have two kinds of approximations $\hat{\mathrm{T}}'^{\alpha}$ and $\hat{\mathrm{T}}''^{\alpha}$ of the same quantities $\hat{\mathrm{t}}^{\alpha}$, differing only in number of terms remaining in the series.

If we substitute expressions (4.3) into formulas (5.5), (5.6), we obtain the final form for approximations of the stresses:

$$\hat{\mathrm{T}}'^{\alpha} = \sqrt{a}h\sum_{k=0}^{N+1}\frac{1+2k}{2}(\overset{(k)}{M}{}^{\alpha\gamma}\mathrm{a}_{\gamma} + \overset{(k)}{Q}{}^{\alpha}\mathrm{n})P_k,$$
$$\hat{\mathrm{T}}''^{\alpha} = \sqrt{a}h\sum_{k=0}^{N}\frac{1+2k}{2}(\overset{(k)}{M}{}^{\alpha\gamma}\mathrm{a}_{\gamma} + \overset{(k)}{Q}{}^{\alpha}\mathrm{n})P_k,$$
$$\hat{\mathrm{T}}^{3} = \sqrt{a}h(\sum_{k=0}^{N+2}\frac{1+2k}{2}\overset{(k)}{M}{}^{3\gamma}\mathrm{a}_{\gamma}P_k + \sum_{k=0}^{N+1}\frac{1+2k}{2}\overset{(k)}{Q}{}^{3}\mathrm{n}P_k).$$

6. Approximations of Deformations and Displacements. Consider an arbitrary displacement vector **u**, which satisfies the conditions (3.6) at a boundary S_u. For simplicity we restrict ourselves to the case of zeroth volume forces (F = 0). It follows from (5.5) that

$$\int_{V_\xi}\{(\hat{\mathrm{T}}'^{i}_{,i}\times\mathrm{n})\cdot(\mathrm{u}\times\mathrm{n}) + (\hat{\mathrm{T}}''^{i}_{,i}\cdot\mathrm{n})(\mathrm{u}\cdot\mathrm{n})\}dV_\xi = 0,$$
$$dV_\xi = d\xi^1 d\xi^2 d\xi^3. \tag{6.1}$$

Integrating (6.1) by parts, we get

$$\int_{V_\xi}\{[\hat{\mathrm{T}}'^{i}\cdot(\mathrm{n}\times(\mathrm{u}\times\mathrm{n}))]_{,i} + [(\hat{\mathrm{T}}''^{i}\cdot\mathrm{n})(\mathrm{u}\cdot\mathrm{n})]_{,i}\}dV_\xi$$
$$= \int_{V_\xi}\{\hat{\mathrm{T}}'^{i}\cdot[\mathrm{n}\times(\mathrm{u}\times\mathrm{n})]_{,i} + \hat{\mathrm{T}}''^{i}\cdot[\mathrm{n}\cdot(\mathrm{u}\cdot\mathrm{n})]_{,i}\}dV_\xi. \tag{6.2}$$

Transforming the right-hand side of (6.2), we obtain

$$\mathrm{RS} = \int_{V_\xi}\{\hat{\mathrm{T}}'^{i}\cdot[\mathrm{n}\times(\mathrm{u}\times\mathrm{n})]_{,i} + \hat{\mathrm{T}}''^{i}\cdot[\mathrm{n}\cdot(\mathrm{u}\cdot\mathrm{n})]_{,i}\}dV_\xi$$
$$= \int_{V_\xi}\{\hat{\mathrm{T}}'^{\alpha}\cdot[\mathrm{n}\times(\mathrm{u}\times\mathrm{n})]_{,\alpha} + \hat{\mathrm{T}}''^{\alpha}\cdot[\mathrm{n}\cdot(\mathrm{u}\cdot\mathrm{n})]_{,\alpha} + \hat{\mathrm{T}}^{3}\cdot\mathrm{u}_{,3}\}dV_\xi \tag{6.3}$$
$$= \int_{V_\xi}\{\hat{\mathrm{t}}^{\alpha}\cdot[\mathrm{n}\times(\sum_{k=0}^{N+1}[u]^k P_k\times\mathrm{n})]_{,\alpha} + \hat{\mathrm{t}}^{\alpha}\cdot[\mathrm{n}\cdot(\sum_{k=0}^{N}[\mathrm{u}]^k P_k\cdot\mathrm{n})]_{,\alpha} + \hat{\mathrm{t}}^{3}\cdot\mathrm{U}''_{,3}\}dV_\xi = \int_{V_\xi}\{\hat{\mathrm{t}}^{\alpha}\cdot\mathrm{U}'_{,\alpha} + \hat{\mathrm{t}}^{3}\cdot\mathrm{U}''_{,3}\}dV_\xi.$$

Here

$$\mathrm{U}' = \sum_{k=0}^{N+1}(\mathrm{n}\times([\mathrm{u}]^k\times\mathrm{n}))P_k + \sum_{k=0}^{N}(\mathrm{n}\cdot([\mathrm{u}]^k\cdot\mathrm{n}))P_k;$$
$$\mathrm{U}'' = \sum_{k=0}^{N+3}(\mathrm{n}\times([\mathrm{u}]^k\times\mathrm{n}))P_k + \sum_{k=0}^{N+2}(\mathrm{n}\cdot([\mathrm{u}]^k\cdot\mathrm{n}))P_k).$$

Substituting for $\hat{\mathrm{t}}^{i}$ in (6.3) the expressions (3.1) and using the symmetry of the stress tensor, we apply further the transformation

$$\mathrm{RS} = \int_{V_\xi}\{\hat{\mathrm{t}}^{\alpha}\cdot\mathrm{U}'_{,\alpha} + \hat{\mathrm{t}}^{3}\cdot\mathrm{U}''_{,3}\}\,dV_\xi = \int_{V}\{\sigma^{\alpha k}\mathrm{э}_k\cdot\mathrm{U}'_{,\alpha} + \sigma^{3k}\mathrm{э}_k\cdot\mathrm{U}''_{,3}\}dV$$
$$= \int_{V}\{\sigma^{\alpha\beta}0{,}5[\mathrm{э}_\beta\cdot\mathrm{U}'_{,\alpha} + \mathrm{э}_\alpha\cdot\mathrm{U}'_{,\beta}] + \sigma^{3\alpha}[\mathrm{э}_\alpha\cdot\mathrm{U}''_{,3} + \mathrm{э}_3\cdot\mathrm{U}'_{,\alpha}] + \sigma^{33}[\mathrm{э}_3\cdot\mathrm{U}''_{,3}]\}dV$$
$$(dV = J d\xi^1 d\xi^2 d\xi^3).$$

Denoting the expressions in square bracket by E_{ij},

$$2E_{\alpha\beta} = \ni_\beta \cdot U'_{,\alpha} + \ni_\alpha \cdot U'_{,\beta},\ 2E_{3\alpha} = \ni_\alpha \cdot U''_{,3} + \ni_3 \cdot U'_{,\alpha}, \quad E_{33} = \ni_3 \cdot U''_{,3}, \tag{6.4}$$

we finally obtain

$$\mathrm{RS} = \int_V \sigma^{ij} E_{ij} dV. \tag{6.5}$$

When comparing (6.4) with the expressions for strains in terms of displacements (3.3), we assume that the quantities E_{ij} are approximations of the strains ε_{ij} as segments of the Legendre's polynomial series, and the vectors $\mathbf{U}'$ and $\mathbf{U}''$ represent, respectively, two approximations of the displacement vector $\mathbf{u}$: one corresponds to the derivatives with respect to the ξ^α-axis, and the other, to the derivatives with respect to the ξ^3-axis.

7. Approximations of Boundary Conditions. Integrate the left-hand side of (6.2); we obtain

$$\begin{aligned} \mathrm{LS} = & \int_\Sigma \{\dot{\mathbf{T}}'^1 \cdot (\mathbf{n} \times (\mathbf{u} \times \mathbf{n})) + (\dot{\mathbf{T}}''^1 \cdot \mathbf{n})(\mathbf{u} \cdot \mathbf{n})\} d\xi^2 d\xi^3 \\ & + \int_\Sigma \{\dot{\mathbf{T}}'^2 \cdot (\mathbf{n} \times (\mathbf{u} \times \mathbf{n})) + (\dot{\mathbf{T}}''^2 \cdot \mathbf{n})(\mathbf{u} \cdot \mathbf{n})\} d\xi^1 d\xi^3 \\ & + \int_{S^+} \dot{\mathbf{T}}^3 \cdot \mathbf{u}\, d\xi^1 d\xi^2 - \int_{S^-} \dot{\mathbf{T}}^3 \cdot \mathbf{u}\, d\xi^1 d\xi^2. \end{aligned} \tag{7.1}$$

We estimate the sum of the first two integrals from (7.1). For this purpose, using the orthogonality of Legendre's polynomials we substitute an appropriate segment of the series $\mathbf{U}'$ for the vector $\mathbf{u}$. Next, since Σ is a lined surface, the following equalities hold:

$$d\xi^1 d\xi^3 = \nu_2^0 / J^0 d\sigma^0,\ d\xi^2 d\xi^3 = \nu_1^0 / J^0 d\sigma^0.$$

Here ν_α^0 are cosines of an outward normal to the side surface Σ at the points of the boundary L; $J^0 = \overset{0}{\ni}_1 \cdot (\overset{0}{\ni}_2 \times \overset{0}{\ni}_3)$; $d\sigma^0 = |d\mathbf{L} \times \ni_3| d\xi^3$; $d\mathbf{L}$ is an increment of the unit vector tangent to the curve L, when moving counterclockwise along the boundary. For the sum of the first two integrals from (7.1) we obtain

$$\int_\Sigma \dot{\mathbf{T}}^\alpha \cdot \mathbf{U}' \nu_\alpha^0 / J^0 d\sigma^0$$

$$(\dot{T}^\alpha = \mathbf{n} \times (\dot{\mathbf{T}}'^\alpha \times \mathbf{n}) + \mathbf{n} \cdot (\dot{\mathbf{T}}''^\alpha \cdot \mathbf{n})).$$

In the last two integrals from (7.1) related to the face surfaces S^+ and S^-, we replace the product $d\xi^1 d\xi^2$ according to the formula

$$d\xi^1 d\xi^2 = \left[\frac{\nu_3 dS}{J}\right]^+ = -\left[\frac{\nu_3 dS}{J}\right]^-,$$

where $\nu_3 = \nu \cdot \ni_3$; ν is an outward normal to the surface S; the plus and minus signs correspond to the surfaces S^+ and S^-.

By the above transformations the equality (7.1) is represented as

$$\mathrm{LS} = \int_\Sigma \frac{\dot{\mathbf{T}}^\alpha \cdot \mathbf{U}'}{J^0} \nu_\alpha^0 d\sigma^0 + \int_{S^+} \frac{\dot{\mathbf{T}}^3 \cdot \mathbf{u}}{J} \nu_3 dS^+ + \int_{S^-} \frac{\dot{\mathbf{T}}^3 \cdot \mathbf{u}}{J} \nu_3 dS^-. \tag{7.2}$$

An approximation of the boundary conditions (3.6), (3.7) by segments of series clearly follows from the first integral from (7.2):

$$\mathbf{U}'|_{\Sigma_u} = \mathbf{u}'_*; \tag{7.3}$$

$$\frac{\dot{\mathbf{T}}^\alpha \nu_\alpha^0}{J^0}\Big|_{\Sigma_\sigma} = \mathbf{P}'_* \ (\Sigma_u \cup \Sigma_\sigma = \Sigma). \tag{7.4}$$

Here

$$\mathbf{u}'_* = \sum_{k=0}^{N+1} (\mathbf{n} \times ([\mathbf{u}_*]^k \times \mathbf{n}))P_k + \sum_{k=0}^{N} (\mathbf{n} \cdot ([\mathbf{u}_*]^k \cdot \mathbf{n}))P_k;$$

$$\mathbf{P}'_* = \sum_{k=0}^{N+1} (\mathbf{n} \times ([\mathbf{P}_*]^k \times \mathbf{n}))P_k + \sum_{k=0}^{N} (\mathbf{n} \cdot ([\mathbf{P}_*]^k \cdot \mathbf{n}))P_k.$$

Next, consider the face surfaces S^+ and S^-. According to the boundary conditions (3.6), (3.7) the displacements

$$\mathbf{u}|_{S_u^+} = \mathbf{u}_*,\ \mathbf{u}|_{S_u^-} = \mathbf{u}_*,$$

are specified on a part of the surfaces S_u^+ and S_u^-, and the stresses

$$\mathbf{t}^3\nu_3|_{S_\sigma^+} = \mathbf{P}_*,\ \mathbf{t}^3\nu_3|_{S_\sigma^-} = \mathbf{P}_*. \tag{7.5}$$

are specified on S_σ^+ and S_σ^-. In the last two integrals of the right-hand side of (7.2) we refer to the quantity $\hat{\mathbf{T}}^3\nu_3/J$ as a surface force. So it is a natural requirement to replace (7.5) on S_σ^+ and S_σ^- by the boundary conditions

$$\frac{\hat{\mathbf{T}}^3\nu_3}{J}\Big|_{S_\sigma^+} = \mathbf{P}_*,\ \frac{\hat{\mathbf{T}}^3\nu_3}{J}\Big|_{S_\sigma^-} = \mathbf{P}_*. \tag{7.6}$$

Since the vector $\mathbf{u}$ is arbitrarily selected, we require the boundary conditions

$$\mathbf{U}''|_{S_u^+} = \mathbf{u}_*,\ \mathbf{U}''|_{S_u^-} = \mathbf{u}_*. \tag{7.7}$$

to be fulfilled on the face surfaces S_u^+ and S_u^-. Finally, with regard to (7.3), (7.4), (7.6), (7.7) the equality (7.2) is written as

$$\mathrm{LS} = \int_{\Sigma_\sigma} \mathbf{P}'_* \cdot \mathbf{U}' d\sigma^0 + \int_{\Sigma_u} \hat{\mathbf{T}}^\alpha \cdot \frac{\mathbf{U}'_*}{J^0}\nu_\alpha^0 d\sigma^0 + \int_{S_\sigma^+} \mathbf{P}_* \cdot \mathbf{U}'' dS^+ + \int_{S_\sigma^-} \mathbf{P}_* \cdot \mathbf{U}'' dS^- + \int_{S_u^+} \frac{\hat{\mathbf{T}}^3 \cdot \mathbf{u}_*}{J}\nu_3 dS^+ + \int_{S_u^-} \frac{\hat{\mathbf{T}}^3 \cdot u_*}{J}\nu_3 dS^-. \tag{7.8}$$

Correlating the left-hand side of the equality (6.2) with its right-hand side derived from formulas (6.5) and (7.8), we obtain

$$\int_V \sigma^{ij} E_{ij} dV = \int_{\Sigma_\sigma} \mathbf{P}'_* \cdot \mathbf{U}' d\sigma^0 + \int_{\Sigma_u} \frac{\hat{\mathbf{T}}^\alpha \mathbf{U}'_*}{J^0}\nu_\alpha^0 d\sigma^0 + \int_{S_\sigma^+} \mathbf{P}_* \cdot \mathbf{U}'' dS^+ + \int_{S_\sigma^-} \mathbf{P}_* \cdot \mathbf{U}'' dS^- + \int_{S_u^+} \frac{\hat{\mathbf{T}}^3 \cdot \mathbf{u}_*}{J}\nu_3 dS^+ + \int_{S_u^-} \frac{\hat{\mathbf{T}}^3 \cdot \mathbf{u}_*}{J}\nu_3 dS^-. \tag{7.9}$$

The relationship (7.9) is a condition for equality (balance) of the work of the external and internal forces.

8. Hook's Law Approximation. We approximate Hook's law (3.4) by the relations

$$\sigma^{ij} = C^{ijks} E_{ks}, \tag{8.1}$$

where E_{ks} are approximations of the strain tensor ε_{ks} (6.4):

$$2E_{\alpha\beta} = \mathbf{э}_\beta \cdot \mathbf{U}'_{,\alpha} + \mathbf{э}_\alpha \cdot \mathbf{U}'_{,\beta},\ 2E_{3\alpha} = \mathbf{э}_\alpha \cdot \mathbf{U}''_{,3} + \mathbf{э}_3 \cdot \mathbf{U}'_{,\alpha},$$
$$E_{33} = \mathbf{э}_3 \cdot \mathbf{U}''_{,3}.$$

We represent (8.1) in a vector form similar to the equalities (3.5):

$$\hat{\mathbf{t}}^i = J\sigma^{ij}\mathbf{э}_j = J(\tilde{C}^{i\alpha} \cdot \mathbf{U}'_{,\alpha} + \tilde{C}^{i3} \cdot \mathbf{U}''_{,3}).$$

Thus, for the coefficients of the series (5.4), (5.6) we get

$$[\hat{t}^i]^k = \frac{1+2k}{2}\int_{-1}^{1} J(\bar{C}^{i\alpha}\cdot \mathbf{U}'_{,\alpha} + \bar{C}^{i3}\cdot \mathbf{U}''_{,3})P_k d\xi^3. \tag{8.2}$$

9. A System of Equations for N-Approximation. Based on the above results we write a system of two-dimensional equations. The lengths of the corresponding segments of the series are specified by the number N. So from here on the system of two-dimensional equations is called the N-approximation of the initial three-dimensional problem of the theory of elasticity.

The two-dimensional system of equations for the N-approximation consists of:

1) The equilibrium equations (5.5) in compact form:

$$\mathbf{n}\times(\hat{\mathbf{T}}'^{i}_{,i}\times\mathbf{n}) + \mathbf{n}\cdot(\hat{\mathbf{T}}''^{i}_{,i}\cdot\mathbf{n}) = 0; \tag{9.1}$$

2) Hooke's law equations (8.2) in the form of the series (5.4):

$$\begin{aligned}
\hat{\mathbf{T}}'^{\alpha} &= \sum_{k=0}^{N+1} P_k \frac{1+2k}{2}\int_{-1}^{1} J(\bar{C}^{\alpha\beta}\cdot \mathbf{U}'_{,\beta} + \bar{C}^{\alpha 3}\cdot \mathbf{U}''_{,3})P_k d\xi^3,\\
\hat{\mathbf{T}}''^{\alpha} &= \sum_{k=0}^{N} P_k \frac{1+2k}{2}\int_{-1}^{1} J(\bar{C}^{\alpha\beta}\cdot \mathbf{U}'_{,\beta} +, \bar{C}^{\alpha 3}\cdot \mathbf{U}''_{,3})P_k d\xi^3,\\
\hat{\mathbf{T}}'^{3} = \hat{\mathbf{T}}''^{3} = \hat{\mathbf{T}}^{3} &= \mathbf{n}\times(\sum_{k=0}^{N+2} P_k \frac{1+2k}{2}\int_{-1}^{1} J(\bar{C}^{3\beta}\cdot \mathbf{U}'_{,\beta} + \bar{C}^{33}\cdot \mathbf{U}''_{,3})\times \mathbf{n}P_k d\xi^3)\\
&+ \mathbf{n}\cdot(\sum_{k=0}^{N+1} P_k \frac{1+2k}{2}\int_{-1}^{1} J(\bar{C}^{3\beta}\cdot \mathbf{U}'_{,\beta} + \bar{C}^{33}\cdot \mathbf{U}''_{,3})\cdot \mathbf{n}P_k d\xi^3);
\end{aligned} \tag{9.2}$$

3) The conditions on the face surfaces S^+ and S^- [formulas (7.6), (7.7)]:

$$\mathbf{U}''|_{S_u^+} = \mathbf{u}_*,\ \mathbf{U}''|_{S_u^-} = \mathbf{u}_*,\ \frac{\hat{\mathbf{T}}^3\nu_3}{J}|_{S_\sigma^+} = \mathbf{P}_*,\ \frac{\hat{\mathbf{T}}^3\nu_3}{J}|_{S_\sigma^-} = \mathbf{P}_*. \tag{9.3}$$

To determine the differential order of the system of equations in partial derivatives (9.1)-(9.3) our reasoning is as follows. The coefficients of the series $\mathbf{U}'$ enter into the strain-displacement relations (6.4) together with their partial derivatives of the first order with respect to the Gaussian coordinates ξ^α on the middle surface S^0, and the coefficients of the series ($\mathbf{U}''$ − $\mathbf{U}'$) occur without derivatives. Correspondingly, the first group of unknown coefficients is called basic, and the second group is called complementary.

The complementary unknowns are found from Eqs. (9.3), which are the boundary conditions on the face surfaces. These equations represent a system of algebraic equations in complementary unknown, whose solution is obtained by expressing the complementary unknowns in terms of the basic unknowns.

Furthermore, if we insert these expressions into (9.2), then we get formulas connecting the vector functions $\hat{\mathbf{T}}'^{\alpha}$, $\hat{\mathbf{T}}''^{\alpha}$, $\hat{\mathbf{T}}^3$ and the basic unknowns which are the coefficients of the series $\mathbf{U}'$. These formulas are linear forms with respect to the coefficients of the series $\mathbf{U}'$ and their first derivatives.

By inserting the expressions for $\hat{\mathbf{T}}'^{\alpha}$, $\hat{\mathbf{T}}''^{\alpha}$, $\hat{\mathbf{T}}^3$ into the equilibrium equations (9.1) we obtain a system of 2(N + 2) + N + 1 scalar equations, each containing 2(N + 2) + N + 1 scalar functions ($\mathbf{n}\times([\mathbf{u}]^k\times\mathbf{n})$ ($k = \overline{0, N+1}$), $([\mathbf{u}]^k, \mathbf{n})$ ($k = \overline{0, N}$)) together with their partial derivatives up to and including the second order. Thus we have a system of 2n order to determine n functions, where

$$n = 2(N+2) + N + 1. \tag{9.4}$$

Our special note is that the differential order of the system for N-approximation does not depend on the form of the boundary conditions on the face surfaces: both stresses and displacements may be specified.

When N = O we get the first approximation. In this case it follows from (9.4) that n = 5, i.e., the number of basic unknowns is equal to five: three displacements of the middle surface and two rotation angles. The corresponding differential order of the system (9.1)-(9.3) is equal to 10.

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